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Asymptotics for the Generalized Two-Dimensional Ginzburg–Landau Equation[1]

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In this paper, the authors have studied a generalized Ginzburg–Landau equation in two spatial dimensions (2D). They have shown that this equation, under periodic boundary conditions, has the maximal attractor with finite Hausdorff dimension. This rigorously establishes the foundation for further investigation of this type of model. © 2000 Academic Press

Key Words: modulation equation; 2D Ginzburg–Landau equation; maximal attractor; Hausdorff dimension.

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1. INTRODUCTION

Many nonlinear dynamical systems are too complicated to be effectively analyzed at the present time. Simplified models are often used to investigate specific mechanisms expected to be important. The Ginzburg–Landau type modulation equations are such simplified mathematical models for nonlinear systems in mechanics, physics, and other areas. Much research exists on long-time dynamics of the cubic Ginzburg–Landau equation in one spatial dimension [1–3, 23–25] and in two spatial dimensions [1, 3, 4, 7]

$$u_t - (1 + i\nu)\Delta u + (1 + i\mu)|u|^2 u - au = 0. \tag{1}$$

More recently, we have studied the dynamical behavior of the generalized Ginzburg–Landau equations in one spatial dimension [9–15].

In this paper, we consider the following generalized Ginzburg–Landau equation in two spatial dimensions (2D), derived by Doelman [8] in the modeling of amplitude evolution in fluids near bifurcation points,

$$u_t = \alpha_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u_x + \alpha_3 |u|^2 u_y + \alpha_4 u^2 \bar{u}_x + \alpha_5 u^2 \bar{u}_y - \alpha_6 |u|^{2\sigma} u, \tag{2}$$

where $\alpha_0 > 0$, $\alpha_j = a_j + ib_j$, $j = 1, \dots, 6$, $a_1 > 0$, $a_6 > 0$, $\sigma > 0$, $t > 0$, $(x, y) \in \Omega$, with the suitable initial condition

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \Omega, \tag{3}$$

and the periodic boundary condition

$$u \text{ is } \Omega\text{-periodic}, \tag{4}$$

where $\Omega = (0, L_1) \times (0, L_2)$. Higher order terms like $|u|^{2\sigma}u$ appear in the derivation of more accurate model equations.

Guo and Wang [16] have considered the attractors for a simpler version of the above Eq. (2) with $\sigma \geq 3$. We have discussed [17] the initial value problem for the above generalized Eq. (2) in unbounded domains, under the following assumption (A1) on σ:

(A1) If $b_1 b_6 > 0$, then we assume that $\sigma \geq (1 + \sqrt{10})/2$; if $b_6 = 0$ or $b_1 b_6 < 0$, then we assume that there exists a positive number $\delta > 0$ such that

$$\frac{1}{\sqrt{1 + (b_1\delta - b_6)^2/(a_1\delta + a_6)^2} - 1} \geq \sigma \geq \frac{1 + \sqrt{10}}{2}. \tag{5}$$

The main result of this paper is the existence of the maximal attractor for the initial-boundary value problem of the generalized 2D Ginzburg–Landau Eq. (2), under the assumption (A2) below:

(A2) If $b_1 b_6 > 0$, then we assume that $\sigma > 2$; if $b_6 = 0$ or $b_1 b_6 < 0$, then we assume that there exists a positive number $\delta > 0$ such that

$$\frac{1}{\sqrt{1 + (b_1\delta - b_6)^2/(a_1\delta + a_6)^2} - 1} > \sigma > 2. \tag{6}$$

The main idea or the difficult part comparing this with our earlier work [17] is the more accurate and more detailed estimate on $\int |u|^{2\sigma} |\nabla u|^2$ for $\sigma > 2$. In our previous work [17], we were unable to get these good estimates. Moreover, in this paper, we perform detailed analysis on the dependence of the dimension of the maximal attractor on the parameters in the equation.

2. EXISTENCE OF THE LOCAL SOLUTION

In this section, we discuss the local existence of the initial-boundary value problem (2)–(4). In the following, $H = L^2(\Omega)$ and $H^k = H^k(\Omega)$ denote the usual Sobolev spaces. We need the Gagliardo–Nirenberg inequality [19] in two dimensions

$$\|\nabla^j u\|_p \le c \|\nabla^m u\|_r^a \|u\|_q^{1-a}, \tag{7}$$

where

$$\frac{1}{p} = \frac{j}{2} + a\left(\frac{1}{r} - \frac{m}{2}\right) + \frac{1-a}{q}$$

with $1 \le q, r \le \infty$. Two further restrictions are $0 \le j < m$ and $\frac{j}{m} \le a < 1$.

We define a linear operator $\mathscr{A}u = -(\mu + (a_1 + ib_1)\Delta u)$, with μ to be chosen subsequently, and let

$$G(u) = (\alpha_0 - \mu)u + \alpha_2 |u|^2 u_x + \alpha_3 |u|^2 u_y + \alpha_4 u^2 \bar{u}_x + \alpha_5 u^2 \bar{u}_y - \alpha_6 |u|^{2\sigma} u.$$

Then, by the results of [18], we know $\mathscr{A}$ is strongly elliptic and $-\mathscr{A}$ generates an analytic semigroup of contractions on H. Also, observe that, for any $a_1 > 0, b_1$ and a suitable μ, the eigenvalue problem $\mu f + (a_1 + ib_1)\Delta f = \lambda f$ with periodic boundary conditions does not have 0 as an eigenvalue. Hence 0 lies in the resolvent set of $-\mathscr{A}$ and we can define the fractional power $\mathscr{A}^{\beta}$ with the domain of definition $D(\mathscr{A}^{\beta}) = X^{\beta}$, for $0 \le \beta \le 1$. Now $X_{\beta} = D(\mathscr{A}^{\beta})$ is a dense subspace of H with graph norm; in particular $X_{1/2} = H^1_{per}$, $D(\mathscr{A}) = X_1 = H^2_{per}$, with graph norm on $X_{1/2}, X_1$ equivalent to usual norms on H^1_{per}, H^2_{per}, respectively.

Thus Eq. (2) can be rewritten as

$$u_t = \mathscr{A}u + G(u).$$

By using (7), we can check that the nonlinear map $G: D(\mathscr{A}^{\beta}) \to H$ is locally Lipschitz for any $\beta > \frac{1}{2}$.

Now, by the local existence results in [18, 20], we get

THEOREM 2.1 (Local Existence). *If $u_0 \in D(\mathscr{A}^{\beta})$, $\beta > \frac{1}{2}$, then there exists a unique solution of the initial-boundary value problem* (2)–(4) *on a finite time interval* $[0, t)$, *and this local solution can be extended uniquely to a maximal interval of existence* $[0, T^*)$. *Moreover, if $T^* < +\infty$, then*

$$\lim_{t \to T^*} \| u(t) \|_{\beta} = +\infty,$$

where $\| \cdot \|_{\beta}$ is the norm of X^{β}. By the regularity theory (*see* [21]), *we know that if $u_0 \in D(\mathscr{A})$, then*

$$u \in C^1([0, T^*); H) \cap C([0, T^*); D(\mathscr{A})).$$

Furthermore, if $T^* < +\infty$, then

$$\lim_{t \to T^*} \| u(t) \|_{H^2} = +\infty.$$

3. A PRIORI ESTIMATES AND GLOBAL EXISTENCE

In order to show that the solution exists for all $t > 0$, we only need some conditions such that

$$\|u\|_{H^2} < \infty, \qquad \text{for all } t > 0. \tag{8}$$

This can be achieved by the following a priori estimates, i.e., $\|u(t)\|_{H^2} < K_1(T, u_0)$, $t \in [0, T]$, $T > 0$. This means $\|u(t)\|_{H^2}$ cannot go to infinity at any finite time. In the following, $\int = \int_{\Omega} dx\,dy$, and $\| \cdot \|_B$ denotes the norm in a Banach space B. In order to establish (8), we derive a priori estimates for the solution of (2)–(4) in the following lemmas.

LEMMA 3.1. *Assume that* $u_0 \in L^2_{per}(\Omega)$ *and* $\sigma > 2$. *Then for the solution* $u(t)$ *of problem* (2)–(4) *we have*

$$\|u\|_2^2 \le \|u_0\|_2^2 e^{-a_6 t} + \frac{K}{a_6}(1 - e^{-a_6 t}) \equiv K_1, \qquad \forall t \ge 0, \tag{9}$$

$$\int_s^t \|\nabla u\|_2^2 \le \|u(s)\|^2 + \frac{K_0}{a_1}(t-s), \tag{10}$$

$$\int_s^t \|u\|_{2\sigma+2}^{2\sigma+1} \le \|u(s)\|^2 + \frac{K_0}{a_6}(t-s), \tag{11}$$

where K, K_0 *can be seen in the proof which is a constant depending on* σ, a_0, a_1, a_6, b_j ($j = 2, 3, 4, 5$), Ω.

Proof. Taking the real part of the L^2-inner product of Eq. (2) with u, we obtain

$$\begin{aligned}\frac{d}{dt}\|u\|_2^2 &= 2a_0\|u\|_2^2 - 2a_1\|\nabla u\|_2^2 - 2a_6\int |u|^{2\sigma+2} \\ &\quad + 2(b_2 - b_4)\mathrm{Im}\left[\int |u|^2 \bar{u}_x u\right] + 2(b_3 - b_5)\mathrm{Im}\left[\int |u|^2 \bar{u}_y u\right] \\ &\le 2a_0\|u\|_2^2 - 2a_1\|\nabla u\|_2^2 - 2a_6\int |u|^{2\sigma+2} \\ &\quad + 2|b_2 - b_4|\int |u|^3 |u_x| + 2|b_3 - b_5|\int |u|^3 |u_y|. \end{aligned}\tag{12}$$

Since

$$2|b_2 - b_4|\int |u|^3|u_x| \le 2|b_2 - b_4|\left(\int |u|^6\right)^{\frac{1}{2}}\left(\int |u_x|^2\right)^{\frac{1}{2}},$$

$$2|b_3 - b_5|\int |u|^3 u_y| \le 2|b_2 - b_4|\left(\int |u|^6\right)^{\frac{1}{2}}\left(\int |u_y|^2\right)^{\frac{1}{2}},$$

we get

$$\begin{aligned}&2|b_2 - b_4|\int |u|^3|u_x| + 2|b_3 - b_5|\int |u|^3|u_y| \\ &\quad \le \sqrt{2}M_1\left(\int |u|^6\right)^{\frac{1}{2}}\left(\int |\nabla u|^2\right)^{\frac{1}{2}} \\ &\quad \le a_1\|\nabla u\|^2 + \frac{M_1^2}{2a_1}\int |u|^6, \qquad \text{where } M_1 = \max\{|b_2 - b_4|, |b_3 - b_5|\}.\end{aligned}$$

Due to Young's inequality ($ab \le \frac{\epsilon}{p}a^p + (1/q\epsilon^{q/p})b^q$, $\forall a, b, \epsilon > 0$, $\forall p, 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$),

$$\frac{M_1^2}{2a_1}\int |u|^6 \le \frac{a_6}{2}\int |u|^{2\sigma+2} + \frac{\sigma-2}{\sigma+1}\left(\frac{6}{a_6(\sigma+1)}\right)^{(\sigma-2)/3}\left(\frac{M_1^2}{2a_1}\right)^{(\sigma+1)/(\sigma-2)}|\Omega|,$$

where $|\Omega|$ denotes the area of Ω, we obtain

$$\begin{aligned}2|b_2 - b_4|\int |u|^3|u_x| &+ 2|b_3 - b_5|\int |u|^3|u_y| \\ &\le a_1\|\nabla u\|^2 + \frac{a_6}{2}\int |u|^{2\sigma+2} + \frac{\sigma-2}{\sigma+1}\left(\frac{6}{a_6(\sigma+1)}\right)^{(\sigma-2)/3} \\ &\quad\times\left(\frac{M_1^2}{2a_1}\right)^{(\sigma+1)/(\sigma-2)}|\Omega|,\end{aligned} \tag{13}$$

and

$$a_0\|u\|^2 \le \frac{a_6}{2}\int |u|^{2\sigma+2} + \frac{\sigma}{\sigma+1}\left(\frac{2}{a_6(\sigma+1)}\right)^{1/\sigma}(a_0)^{(\sigma+1)/\sigma}|\Omega|. \tag{14}$$

Combining (12) with (13), (14) we get

$$\frac{d}{dt}\|u\|_2^2 + a_1\|\nabla u\|_2^2 + a_6\int |u|^{2\sigma+2} \le K_0, \tag{15}$$

where

$$\begin{aligned}K_0 = &\frac{\sigma-2}{\sigma+1}\left(\frac{6}{a_6(\sigma+1)}\right)^{(\sigma-2)/3}\left(\frac{M_1^2}{2a_1}\right)^{(\sigma+1)/(\sigma-2)}|\Omega| \\ &+ \frac{\sigma}{\sigma+1}\left(\frac{2}{a_6(\sigma+1)}\right)^{1/\sigma}(a_0)^{(\sigma+1)/\sigma}\Big)|\Omega|.\end{aligned}$$

In view of the Young's inequality, we deduce that

$$\|u\|_2^2 \le \int |u|^{2\sigma+2} + \frac{\sigma}{(\sigma+1)}^{(\sigma+1)/\sigma}|\Omega|. \tag{16}$$

By (15) and (16), we infer that

$$\frac{d}{dt}\|u\|_2^2 + a_6\|u\|_2^2 \le K, \tag{17}$$

where

$$K = \frac{\sigma - 2}{\sigma + 1} \frac{6}{a_6(\sigma + 1)}^{(\sigma-2)/3}$$
$$\times \left(\left(\left(\frac{M_1^2}{2a_1} \right)^{(\sigma+1)/(\sigma-2)} |\Omega| + \left(\frac{\sigma}{\sigma + 1} \left(\frac{2}{a_6(\sigma + 1)} \right)^{1/\sigma} \right.\right.\right.$$
$$\left.\left. \times (a_0)^{(\sigma+1)/\sigma} + a_6 \frac{\sigma}{(\sigma + 1)} \right)^{(\sigma+1)/\sigma} \right) |\Omega|.$$

Thus by the Gronwall inequality [22], we have

$$\|u\|_2^2 \leq \|u_0\|_2^2 e^{-a_6 t} + \frac{K}{a_6}(1 - e^{-a_6 t}), \qquad \forall t \geq 0. \tag{18}$$

So, the estimate (9) is proved. By integrating (15), we obtain (10), (11). This completes the proof of Lemma 3.1.

LEMMA 3.2. *Under the assumption of Lemma* 3.1 *and* $u_0 \in H_{per}^1$, *we have the following estimates for the solution* $u(t)$ *of problem* (2)–(4),

$$\begin{aligned}
\frac{1}{2(1 + \sigma)} \frac{d}{dt} \int |u|^{2\sigma+2}
&\leq a_0 \int |u|^{2\sigma+2} - \frac{a_6}{2} \int |u|^{4\sigma+2} \\
&\quad + \frac{12}{a_6} |\alpha_2|^2 \left(\epsilon_1 \int |u|^{2\sigma} |\nabla u|^2 + \frac{\sigma - 2}{\sigma \epsilon_1^{2/(\sigma-2)}} \|\nabla u\|^2 \right) \\
&\quad - \frac{1}{4} \int |u|^{2\sigma-2} \big(a_1(1 + 2\sigma) \|\nabla |u|^2|^2 - 2b_1 \sigma \nabla |u|^2 \\
&\quad \cdot i(u \nabla \bar{u} - \bar{u} \nabla u) + a_1 |u \nabla \bar{u} - \bar{u} \nabla u|^2 \big),
\end{aligned} \tag{19}$$

where ϵ_1 *is some positive constant.*

Proof. This can be proved as follows. Taking the inner product in H of (2) with $|u|^{2\sigma} u$, we find that

$$\begin{aligned}
\int u_t |u|^{2\sigma} \bar{u} &= a_0 \int |u|^{2\sigma+2} + \alpha_1 \int \Delta u |u|^{2\sigma} \bar{u} - \alpha_6 \int |u|^{4\sigma+2} \\
&\quad + \alpha_2 \int u_x |u|^{2\sigma+2} \bar{u} + \alpha_3 \int |u|^{2\sigma+2} u_y \bar{u} \\
&\quad + \alpha_4 \int |u|^{2\sigma+2} \bar{u}_x u + \alpha_5 \int |u|^{2\sigma+2} \bar{u}_y u.
\end{aligned} \tag{20}$$

Using the facts (see [4, 5])

$$\alpha_1 \int \Delta u |u|^{2\sigma} \bar{u} = -\alpha_1 \int |\nabla u|^2 |u|^{2\sigma} - \sigma\alpha_1 \int |u|^{2\sigma-2} \bar{u} \nabla u \cdot \nabla |u|^2, \quad (21)$$

and then taking the real part of (20) we get

$$\begin{aligned}
&\frac{1}{2(1+\sigma)} \frac{d}{dt} \int |u|^{2\sigma+2} \\
&\quad = a_0 \int |u|^{2\sigma+2} - a_1 \int |\nabla u|^2 |u|^{2\sigma} - \frac{a_1 \sigma}{2} \int |u|^{2\sigma-2} |\nabla |u|^2|^2 \\
&\qquad + \frac{b_1 \sigma}{2} \int |u|^{2\sigma-2} \nabla |u|^2 \cdot i(u \nabla \bar{u} - \bar{u} \nabla u) - a_6 \int |u|^{4\sigma+2} \\
&\qquad + \operatorname{Re}\Big\{ \alpha_2 \int u_x |u|^{2\sigma+2} \bar{u} + \alpha_3 \int |u|^{2\sigma+2} u_y \bar{u} \\
&\qquad\qquad + \alpha_4 \int |u|^{2\sigma+2} \bar{u}_x u + \alpha_5 \int |u|^{2\sigma+2} \bar{u}_y u \Big\}. \qquad (22)
\end{aligned}$$

Now using the identity

$$|u|^2 |\nabla u|^2 = \frac{1}{4} \big(|\nabla |u|^2|^2 + |u \nabla \bar{u} - \bar{u} \nabla u|^2 \big), \quad (23)$$

and the Young's inequality, together with the estimate (9), we obtain

$$\begin{aligned}
&-a_1 \int |\nabla u|^2 |u|^{2\sigma} - \frac{a_1 \sigma}{2} \int |u|^{2\sigma-2} |\nabla |u|^2|^2 \\
&\quad + \frac{b_1 \sigma}{2} \int |u|^{2\sigma-2} \nabla |u|^2 \cdot i(u \nabla \bar{u} - \bar{u} \nabla u) \\
&\quad = -\frac{1}{4} \int |u|^{2\sigma-2} \big(a_1 (1+2\sigma) |\nabla |u|^2|^2 - 2 b_1 \sigma \nabla |u|^2 \\
&\qquad\qquad \cdot i(u \nabla \bar{u} - \bar{u} \nabla u) + a_1 |u \nabla \bar{u} - \bar{u} \nabla u|^2 \big), \qquad (24)
\end{aligned}$$

$$\begin{aligned}
a_0 \int |u|^{2\sigma+2} &\le \frac{a_6}{6} \int |u|^{4\sigma+2} + \frac{3 a_0^2}{2 a_6} \|u\|_2^2 \\
&\le \frac{a_6}{6} \int |u|^{4\sigma+2} + \frac{3 a_0^2}{2 a_6} K_1, \qquad \forall t \ge 0. \qquad (25)
\end{aligned}$$

We now estimate the last four terms in (22). We only estimate $\alpha_2 \int |u|^{2\sigma+2} u_x \bar{u}$, as the estimates of the other three terms are similar,

$$\operatorname{Re}\left(\alpha_2 \int |u|^{2\sigma+2} u_x \bar{u} \right) \le |\alpha_2| \int |u_x|\, |u|^2 |u|^{2\sigma+1} \le |\alpha_2| \int |\nabla u|\, |u|^2 |u|^{2\sigma+1}$$
$$\le \frac{a_6}{12} \int |u|^{4\sigma+2} + \frac{3}{a_6} |\alpha_2|^2 \int |\nabla u|^2 |u|^4.$$

Because

$$\int |\nabla u|^2 |u|^4 = \int |u|^4 |\nabla u|^{4/\sigma} |\nabla u|^{(2\sigma-4)/\sigma}$$
$$\le \epsilon_1 \int |u|^{2\sigma} |\nabla u|^2 + \frac{\sigma-2}{\sigma \epsilon_1^{2/(\sigma-2)}} \|\nabla u\|^2,$$

we conclude that

$$\operatorname{Re}\left(\alpha_2 \int |u|^{2\sigma+2} u_x \bar{u} \right)$$
$$\le \frac{a_6}{12} \int |u|^{4\sigma+2}$$
$$+ \frac{3}{a_6} |\alpha_2|^2 \left(\epsilon_1 \int |u|^{2\sigma} |\nabla u|^2 + \frac{\sigma-2}{\sigma \epsilon_1^{2/(\sigma-2)}} \|\nabla u\|^2 \right). \tag{26}$$

Let $(3/a_6)|\alpha_2|^2((\sigma-2)/\sigma\epsilon_1^{2/(\sigma-2)}) \equiv A_1$; here and after, A_i $(i = 1, 2, \dots)$ denotes the constants depending on the parameters in Eq. (2). (For simplicity, if α_2 is replaced by α_3, α_4, and α_5, we also use A_1 to denote these numbers.) By (22), (25), (26) we have

$$\frac{1}{2(1+\sigma)} \frac{d}{dt} \int |u|^{2\sigma+2}$$
$$\le a_0 \int |u|^{2\sigma+2} - \frac{a_6}{2} \int |u|^{4\sigma+2} + \frac{12}{a_6} |\alpha_2|^2 \epsilon_1 \int |u|^{2\sigma} |\nabla u|^2 + 4A_1 \|\nabla u\|^2$$
$$- \frac{1}{4} \int |u|^{2\sigma-2} \big(a_1(1+2\sigma) |\nabla |u|^2|^2 - 2b_1 \sigma \nabla |u|^2$$
$$\cdot i(u\nabla\bar{u} - \bar{u}\nabla u) + a_1 |u\nabla\bar{u} - \bar{u}\nabla u|^2 \big). \tag{27}$$

This completes the proof of Lemma 3.2.

LEMMA 3.3. *Under the assumptions of Lemma* 3.1, *Lemma* 3.2, *and* (A2), *we have*

$$\|\nabla u\|_2 \le K_2(T, K_1, \|u_0\|_{H^1}), \qquad \text{for } 0 \le t \le T.$$

Proof. Taking the real part of the L^2-inner product of (2) with Δu, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 - a_0\|\nabla u\|_2^2 + a_1\|\Delta u\|_2^2 - \mathrm{Re}\left[\alpha_6\int |u|^{2\sigma} u\Delta\bar{u}\right]$$
$$+ \mathrm{Re}\left[\alpha_2\int |u|^2 u_x\Delta\bar{u} + \alpha_3\int |u|^2 u_y\Delta\bar{u}\right]$$
$$+ \mathrm{Re}\left[\alpha_4\int u^2\bar{u}_x\Delta\bar{u} + \alpha_5\int u^2\bar{u}_y\Delta\bar{u}\right] = 0. \tag{28}$$

We now estimate the last three terms in (28). By (21) and (23) we find that

$$\begin{aligned}
&\mathrm{Re}\left[\alpha_6\int |u|^{2\sigma} u\Delta\bar{u}\right]\\
&= -\mathrm{Re}\left[\alpha_6\int |u|^{2\sigma}|\nabla u|^2\right] - \mathrm{Re}\left[\alpha_6\int \sigma|u|^{2\sigma-2}u\nabla\bar{u}\cdot\nabla|u|^2\right]\\
&= -a_6\int |u|^{2\sigma}|\nabla u|^2 - \frac{\sigma a_6}{2}\int |u|^{2\sigma-2}|\nabla|u|^2|^2 + \frac{\sigma b_6}{2}\\
&\quad\times\int |u|^{2\sigma-2}\nabla|u|^2\cdot i(\bar{u}\nabla u - u\nabla\bar{u})\\
&= -\frac{1}{4}\int |u|^{2\sigma-2}\Big(a_6(1+2\sigma)|\nabla|u|^2|^2 - 2b_6\sigma\nabla|u|^2\\
&\qquad \cdot i(\bar{u}\nabla u - u\nabla\bar{u}) + a_6|\bar{u}\nabla u - u\nabla\bar{u}|^2\Big).
\end{aligned} \tag{29}$$

We only estimate the term $\alpha_2\int |u|^2 u_x\Delta\bar{u}$ in (29), as the remaining three terms can be similarly estimated,

$$\begin{aligned}
\alpha_2\int |u|^2 u_x\Delta\bar{u} &\le |\alpha_2|\int |u|^2|\nabla u|\,|\Delta u|\\
&\le \eta\|\Delta u\|_2^2 + \frac{|\alpha_2|^2}{4\eta}\int |u|^4|\nabla u|^2\\
&\le \eta\|\Delta u\|_2^2 + \frac{|\alpha_2|^2}{4\eta}\epsilon_1\int |u|^{2\sigma}|\nabla u|^2 + A_2\|\nabla u\|^2,
\end{aligned} \tag{30}$$

with $(|\alpha_2|^2/4\eta)((\sigma-2)/\sigma\epsilon_1^{2/(\sigma-2)}) \equiv A_2$. (For simplicity, if α_2 is replaced by α_3, α_4, and α_5, we also use A_2 to denote these numbers.) By (30), we conclude that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + (a_1 - 4\eta)\|\Delta u\|_2^2 \le \frac{|\alpha_2|^2}{\eta}\epsilon_1\int|u|^{2\sigma}|\nabla u|^2 + (a_0 + 4A_2)\|\nabla u\|^2$$
$$\frac{1}{4}\int|u|^{2\sigma-2}\big(a_6(1+2\sigma)|\nabla|u|^2|^2 - 2b_6\sigma\nabla|u|^2$$
$$\cdot i(\bar{u}\nabla u - u\nabla\bar{u}) + |\bar{u}\nabla u - u\nabla\bar{u}|^2\big). \tag{31}$$

Furthermore, by (27), (31), and choosing η such that $\eta \le a_1/8$ we find that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_2^2 + \frac{\delta}{1+\sigma}\int|u|^{2\sigma+2}\right) + \frac{a_1}{2}\|\Delta u\|_2^2 + \frac{\delta a_6}{2}\int|u|^{4\sigma+2}$$
$$\le \left(\frac{12\delta}{a_6}|\alpha_2|^2 + 8\frac{|\alpha_2|^2}{a_1}\right)\epsilon_1\int|u|^{2\sigma}|\nabla u|^2 + (a_0 + 4A_2 + 4\delta A_1)\|\nabla u\|^2$$
$$+ \delta a_0\int|u|^{2\sigma+2} - \frac{1}{4}\int|u|^{2\sigma-2}\big((a_1\delta + a_6)(1+2\sigma)|\nabla|u|^2|^2$$
$$-2\sigma(b_1\delta - b_6)\nabla|u|^2\cdot i(\bar{u}\nabla u - u\nabla\bar{u})\big)$$
$$+\frac{1}{4}\int|u|^{2\sigma-2}(a_1\delta + a_6)|\bar{u}\nabla u - u\nabla\bar{u}|^2\big). \tag{32}$$

Note that if the assumption (A2) is satisfied, then $(a_1\delta + a_6)(1 + 2\sigma)|\nabla|u|^2|^2 - 2\sigma(b_1\delta - b_6)\nabla|u|^2\cdot i(\bar{u}\nabla u - u\nabla\bar{u}) + (a_1 + a_6\delta)|\bar{u}\nabla u - u\nabla\bar{u}|^2$ is positive. Now, let λ_σ be the smaller eigenvalue for the matrix $\begin{pmatrix}(a_1+\delta a_6)(1+2\sigma) & -\sigma(b_1\delta - b_6)\\ -\sigma(b_1\delta - b_6) & (a_1 + a_6\delta)\end{pmatrix}$. Due to the condition of (A2), we know that $\lambda_\sigma > 0$. So, we get

$$\frac{1}{4}\int|u|^{2\sigma-2}\big((a_1\delta + a_6)(1+2\sigma)|\nabla|u|^2|^2$$
$$-2\sigma(b_1\delta - b_6)\nabla|u|^2\cdot i(\bar{u}\nabla u - u\nabla\bar{u})\big)$$
$$+\frac{1}{4}\int|u|^{2\sigma-2}(a_1 + a_6\delta)|\bar{u}\nabla u - u\nabla\bar{u}|^2)$$
$$\ge \lambda_\sigma\int|u|^{2\sigma}|\nabla u|^2.$$

We choose ϵ_1 such that $((12\delta/a_6)|\alpha_2|^2 + 8(|\alpha_2|^2/a_1))\epsilon_1 \le \lambda_\sigma$ and (32) can then be rewritten as

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_2^2 + \frac{\delta}{1+\sigma}\int |u|^{2\sigma+2}\right) + \frac{a_1}{2}\|\Delta u\|_2^2 + \frac{\delta a_6}{4}\int |u|^{4\sigma+2}$$
$$\le (a_0 + 4A_2 + 4\delta A_1)\|\nabla u\|^2 + \delta a_0 \int |u|^{2\sigma+2}$$
$$\le A_3\left(\|\nabla u\|^2 + \int |u|^{2\sigma+2}\right), \tag{33}$$

where $A_3 = \max\{a_0 + 4A_2 + 4\delta A_1, \delta a_0\}$. Therefore, by the Gronwall inequality, we get

$$\|\nabla u\|_2^2 + \frac{\delta}{1+\sigma}\int |u|^{2\sigma+2} \le K_2(T, K_1\|u_0\|_{H^1}), \qquad \text{for } 0 \le t \le T, \tag{34}$$

where the embedding $H^1 \hookrightarrow L^p, 2 \le p < \infty$ is used. Thus Lemma 3.3 is proved.

LEMMA 3.4. *Under the assumption of Lemma* 3.1 *and* (A2), *we have*

$$\|\Delta u\|_2 \le K_3(T, K_2, \|u_0\|_{H^2}), \qquad \textit{for } 0 \le t \le T.$$

Proof. Taking the real part of the inner product of (2) with $\Delta^2 u$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 = a_0\|\Delta u\|_2^2 - \|\nabla\Delta u\|_2^2 - \operatorname{Re}\alpha_6\int |u|^{2\sigma}u\Delta^2\bar{u}$$
$$+ \operatorname{Re}\Bigg[\alpha_2\int |u|^2 u_x\Delta^2\bar{u} + \alpha_3\int |u|^2 u_y \Delta^2\bar{u}$$
$$+ \alpha_4\int u^2\bar{u}_x\Delta^2\bar{u} + \alpha_5\int u^2\bar{u}_y\Delta^2\bar{u}. \tag{35}$$

After integration by parts and using Lemma 3.3 and some elementary manipulation, we get

$$\frac{d}{dt}\|\Delta u\|_2^2 \le A_4\|\Delta u\|_2^4 - \|\nabla\Delta u\|_2^2 + A_5, \tag{36}$$

where A_4 depends on $a_0, |\alpha_i|^2$ $(i = 2,\dots,6)$ and A_5 depends on $|\alpha_i|^2$ $(i = 2,\dots,6)$, K_2. Thus, by the Gronwall inequality, we obtain

$$\|\Delta u\|_2 \le K_3(T, K_2, \|u_0\|_{H^2}). \tag{37}$$

By Theorem 2.1, Lemma 3.1, Lemma 3.3, and Lemma 3.4, we finally get the following global existence result.

THEOREM 3.5 (Global Existence). *Under the assumption* $(A2)$ *there exists a unique global solution of the initial-boundary value problem for the* $2D$ *generalized Ginzburg–Landau Eqs*. (2)–(4) *in* $H^2(\Omega)$.

4. THE MAXIMAL ATTRACTOR AND ITS DIMENSION

In the last section, we have established the existence of the global dynamical system (solution map) $S(t)(t \geq 0)$, which maps $H^2(\Omega)$ into $H^2(\Omega)$ via $S(t)u_0 = u(t)$, the solution of problem (2)–(4). By a standard result in [18], we know that $S(t)$ is compact for $t > 0$.

Now, we obtain the absorbing set for $S(t)$ in $H^2(\Omega)$. By Lemma 3.1, we get

$$\|u(t)\|_2^2 \leq \frac{2K}{a_6}, \qquad \text{for } t \geq \frac{1}{a_6}\log\frac{a_6 R_0^2}{K}, \text{ if } \|u_0\|_2^2 \leq R_0. \tag{38}$$

We use the uniform Gronwall inequality [22] to (33) and combine (10), (11), and (38); we obtain

$$\|\nabla u(t)\|_2^2 \leq A_6 \exp\left[2A_3\left(1 + \frac{1+\sigma}{\delta}\right)^2 A_6\right], \qquad \text{for } t \geq 1 + \frac{1}{a_6}\log\frac{a_6 R_0^2}{K}, \tag{39}$$

where $A_6 = 1 + 2K/a_6 + K_0/a_1 + \delta(2K + K_0)/a_6(1+\sigma)$. Similar to the deduction of (39) and using Lemma 3.4, we have

$$\|\Delta u(t)\|_2^2 \leq A_7^2, \qquad \text{for } t \geq 2 + \frac{1}{a_6}\log\frac{a_6 R_0^2}{K}, \tag{40}$$

where

$$\begin{aligned} A_7 &= \left[(A_8 + A_6)\exp(A_4 A_8)\right]^{\frac{1}{2}} \\ A_8 &= \frac{2A_3}{a_1}\left\{A_6 \exp\left[2A_3\left(1 + \frac{1+\sigma}{\delta}\right)^2 A_6\right]\right\}^2 \\ &\quad + A_6 \exp\left[2A_3\left(1 + \frac{1+\sigma}{\delta}\right)^2 A_6\right]. \end{aligned} \tag{41}$$

Therefore, the estimates (38), (39), and (40) show that there exists an absorbing set

$$\mathscr{B} = \left\{ u \in H^2(\Omega), \|u\|_{H^2} \le \rho \right\}, \tag{42}$$

with $\rho = [2K/a_6]^{1/2} + \{A_6 \exp[2A_3(1 + \frac{1+\sigma}{\delta})^2 A_6]\}^{1/2} + A_7$.

By the results of [22], we obtain the following theorem about the existence of the global attractor for $S(t)$.

THEOREM 4.1. *Assume that* (A2) *holds*; *then the* ω*-limit set*

$$\mathscr{A} = \omega(\mathscr{B}) = \bigcap_{s \ge 0} \overline{\bigcup_{t \ge s} S(t)\mathscr{B}}$$

is the maximal, *compact*, *connected attractor for* $S(t)$ *on* $H^2(\Omega)$, *where the closure is taken in* $H^2(\Omega)$.

Next, we will show that the Hausdorff and fractal dimensions of the maximal attractor $\mathscr{A}$ are finite. We rewrite (2) in an abstract form,

$$\frac{du}{dt} = F(u),$$

where $F(u)$ stands for the right hand side of (2).

Consider the first variation equation of problem (2)–(4),

$$v_t = F'(u(t))v \tag{43}$$

with initial condition

$$v(0) = v_0 \in H \tag{44}$$

and v is Ω-periodic, where

$$\begin{aligned} F'(u(t))v = {} & \alpha_0 v + \alpha_1 \Delta v + \alpha_2\left(|u|^2 v_x + \bar{u} u_x v + u u_x \bar{v}\right) \\ & + \alpha_3\left(|u|^2 v_y + \bar{u} u_y v + u u_y \bar{v}\right) + \alpha_4\left(u^2 \bar{v}_x + 2u\bar{u}_x v\right) \\ & + \alpha_5\left(u^2 \bar{v}_x + 2u\bar{u}_x v\right) - \alpha_6\left((1+\sigma)|u|^{2\sigma} v + \sigma |u|^{2\sigma-2} u^2 \bar{v}\right) \end{aligned}$$

and $u(t) = S(t)u_0$ is the solution of (2)–(4) with $u_0 \in \mathscr{A}$.

We know that the solution of (2)–(4), $S(t)u_0 \in H^2(\Omega)$ if $u_0 \in \mathscr{A} \subset H^2(\Omega)$. (In fact, by the regularity of the results of [18], we know $\mathscr{A} \subset C^\infty(\Omega)$.) By standard methods we can show that $\forall v_0 \in H$, the linear initial boundary value problem (43)–(44) possesses a unique solution $v(t)$ such that

$$v(t) \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; H), \qquad \forall T > 0. \tag{45}$$

For $\forall v_0 \in H$, we denote by $G(t)v_0$ the solution of (43)–(44). After a complicated computation, we can find that $\forall w_0, u_0 \in \mathscr{A}$,

$$\frac{\| S(t)w_0 - S(t)u_0 - G(t)(w_0 - u_0)\|_2^2}{\|w_0 - u_0\|_2^2} \le C\|w_0 - u_0\|_2^2, \qquad \forall 0 \le t \le T,$$

where C depends on T and ρ. This inequality shows that the semigroup $S(t)$ is uniformly differentiable on $\mathscr{A}$. The differential in H of $S(t)$ at $u_0 \in \mathscr{A}$ is $L(t, u_0)$: $v_0 \in H \to G(t)v_0 \in H$. For all m, we consider $v_0 = v_{01}, \dots, v_{0m}$, m elements of H, and the corresponding solutions $v(t) = v_1(t), \dots, v_m(t)$ of (43)–(44). Then it follows from [22] that

$$\begin{aligned} &\left| v_1(t) \wedge \cdots \wedge v_m(t) \right|_{\wedge^m H} \\ &\quad = \left| v_{01}(t) \wedge \cdots \wedge v_{0m}(t) \right|_{\wedge^m H} \exp\left(\int_0^t \operatorname{Re} \operatorname{Tr} F'(u(t)) \circ Q_m(\tau)\, d\tau \right), \end{aligned} \tag{46}$$

where $Q_m(\tau) = Q_m(\tau, u_0, v_{01}, \dots, v_{0m})$ is the orthogonal projection in H onto the space spanned by $v_1(\tau), \dots, v_m(\tau)$. At a given time τ, let $\varphi_j(\tau) \in H^1(\Omega)$, $j \in N$, be an orthogonal basis of H with $\varphi_1(\tau), \dots, \varphi_m(\tau)$ spanning $Q_m(\tau)H = \operatorname{span}\{v_1(\tau), \dots, v_m(\tau)\}$. From (45), $v_j(\tau) \in H^1(\tau)$ for a.e. τ. Thus

$$\begin{aligned} \operatorname{Re} \operatorname{Tr} F'(u(\tau)) \circ Q_m(\tau) &= \sum_{j=1}^{\infty} \operatorname{Re}\big(F'(u(\tau))\, \varphi_j(\tau), \varphi_j(\tau) \big) \\ &= \sum_{j=1}^{m} \operatorname{Re}\big(F'(u(\tau))\, \varphi_j(\tau), \varphi_j(\tau) \big). \end{aligned} \tag{47}$$

Omitting temporarily the variable τ, we see that

$$\begin{aligned} &\operatorname{Re}\big(F'(u)\, \varphi_j, \varphi_j \big) \\ &\quad = \alpha_0 \|\varphi_j\|_2^2 - a_1 \|\nabla \varphi_j\|_2^2 - a_6(1+\sigma) \int |u|^{2\sigma} |\varphi_j|^2 \\ &\qquad + \operatorname{Re} \alpha_2 \int |u|^2 \varphi_{jx} \overline{\varphi}_j + 2 \operatorname{Re} \alpha_2 \int u u_x \overline{\varphi}_j^2 \\ &\qquad + \operatorname{Re} \alpha_3 \int |u|^2 \varphi_{jy} \overline{\varphi}_j + 2 \operatorname{Re} \alpha_3 \int u u_y \overline{\varphi}_j^2 \\ &\qquad + \operatorname{Re} \alpha_4 \int u^2 \overline{\varphi}_{jx} \overline{\varphi}_j + 2 \operatorname{Re} \alpha_4 \int u \overline{u}_x |\varphi_j|^2 + \operatorname{Re} \alpha_5 \int u^2 \overline{\varphi}_{jy} \overline{\varphi}_j \\ &\qquad + 2 \operatorname{Re} \alpha_5 \int u \overline{u}_y |\varphi_j|^2 - \operatorname{Re} \sigma \alpha_6 \int |u|^{2\sigma - 2} u^2 \overline{\varphi}_j^2. \end{aligned} \tag{48}$$

We infer that

$$\begin{aligned}
&\operatorname{Re}\alpha_2\int|u|^2\varphi_{jx}\overline{\varphi}_j + 2\operatorname{Re}\alpha_2\int uu_x\overline{\varphi}_j^2\\
&\quad\le|\alpha_2|\,\|u\|_\infty^2\|\nabla\varphi_j\|_2\|\varphi_j\|_2 + 2|\alpha_2|\,\|u\|_\infty\|\nabla u\|_2\|\varphi_j\|_4^2\\
&\quad\le\frac{a_1}{8}\|\nabla\varphi_j\|_2^2 + \frac{4|\alpha_2|^2}{a_1}\left(\|u\|_\infty^4 + 4\|u\|_\infty^2\|\nabla u\|_2^2\right)\|\varphi_j\|_2^2\\
&\quad\le\frac{a_1}{8}\|\nabla\varphi_j\|_2^2 + A_9\|\varphi_j\|_2^2,
\end{aligned}$$

where $A_9 = (4|\alpha_2|^2/a_1)\{2CA_7^2/a_6 + 4CA_6A_7\sqrt{2K/a_6}\,\exp[2A_3(1+\frac{1+\sigma}{\delta})^2A_6]\}$, where C is the constant in (7). (For simplicity, in the above inequality, if α_2 is replaced by α_3, α_4, and α_5, we also use A_9 to denote these numbers.) Furthermore,

$$\begin{aligned}
&\operatorname{Re}\alpha_2\int|u|^2\varphi_{jx}\overline{\varphi}_j + 2\operatorname{Re}\alpha_2\int uu_x\overline{\varphi}_j^2 + \operatorname{Re}\alpha_3\int|u|^2\varphi_{jy}\overline{\varphi}_j + 2\operatorname{Re}\alpha_3\int uu_y\overline{\varphi}_j^2\\
&\quad+\operatorname{Re}\alpha_4\int u^2\overline{\varphi}_{jx}\overline{\varphi}_j + 2\operatorname{Re}\alpha_4\int u\bar{u}_x|\varphi_j|^2 + \operatorname{Re}\alpha_5\int u^2\overline{\varphi}_{jy}\overline{\varphi}_j\\
&\quad+2\operatorname{Re}\alpha_5\int u\bar{u}_y|\varphi_j|^2\\
&\quad\le\frac{a_1}{2}\|\nabla\varphi_j\|_2^2 + 4A_9\|\varphi_j\|_2^2
\end{aligned}\tag{49}$$

$$-\operatorname{Re}\sigma\alpha_6\int|u|^{2\sigma-2}u^2\overline{\varphi}_j^2\le|\sigma\alpha_6|\,\|u\|_\infty^{2\sigma}\|\varphi_j\|_2^2\le A_{10}\|\varphi_j\|_2^2,\tag{50}$$

where $A_{10} = C\sigma|\alpha_6|A_7^\sigma(2K/a_6)^{\sigma/2}$, and C is the constant in (7).

By (47)–(50) we have

$$\operatorname{Re}\operatorname{Tr}F'(u(\tau))\circ Q_m(\tau)\le-\frac{a_1}{2}\sum_{j=1}^m\|\nabla\varphi_j\|_2^2 + A_{11}\sum_{j=1}^m\|\varphi_j\|_2^2,\tag{51}$$

where $A_{11} = 4A_9 + A_{10} + \alpha_0$.

Let $\eta = \eta(x,t) = \sum_{j=1}^m|\varphi_j|^2$. Since $\{\varphi_j\}$ $(j = 1,2,\dots)$ is the orthogonal in H, we infer that

$$\sum_{j=1}^m\|\varphi_j\|_2^2 = \int_\Omega\eta = m.\tag{52}$$

By the Sobolev–Lieb–Thirring inequality [22], we get

$$\int_\Omega \eta^2\,dx \le C_0(\Omega)\int_\Omega \eta\,dx + C_0(\Omega)\sum_{j=1}^{m}\|\nabla\varphi_j\|_2^2. \tag{53}$$

By using the Hölder inequality, we have

$$\left(\int_\Omega \eta\,dx\right)^2 \le |\Omega|\int_\Omega \eta^2\,dx. \tag{54}$$

From (51)–(54) and Young's inequality, we obtain

$$\begin{aligned}\operatorname{Re}\operatorname{Tr} F'(u(\tau))\circ Q_m(\tau) &\le -\frac{a_1}{2C_0|\Omega|}m^2 + \left(\frac{a_1}{2} + A_{11}\right)m\\ &\le -\frac{a_1}{4C_0|\Omega|}m^2 + \frac{C_0|\Omega|}{a_1}\left(\frac{a_1}{2} + A_{11}\right)^2. \end{aligned}\tag{55}$$

For $i = 1, 2, \ldots, m$ and $v_{0i} \in H$, we define

$$\begin{aligned} q_m(t) &= \sup_{u_0\in\mathscr{A}}\sup_{\|v_{0i}\|_2^2\le 1}\left(\frac{1}{t}\int_0^t \operatorname{Re}\operatorname{Tr} F'(S(\tau)u_0\circ Q_m(\tau)\,d\tau\right),\\ q_m &= \lim_{t\to\infty} q_m(t). \end{aligned}$$

It follows from (54) that

$$q_m \le -\frac{a_1}{4C_0|\Omega|}m^2 + \frac{C_0|\Omega|}{a_1}\left(\frac{a_1}{2} + A_{11}\right)^2.$$

This shows that if m is defined by

$$m - 1 < \frac{2C_0|\Omega|}{a_1}\left(\frac{a_1}{2} + A_{11}\right) \le m, \tag{56}$$

then $q_m < 0$ and thus by the results of [22], we obtain the following theorem:

THEOREM 4.2. *Let $\mathscr{A}$ be the global attractor of problem for* (2)–(4), *as obtained in Theorem* 4.1. *Then the Hausdorff dimension of $\mathscr{A}$ is less than or equal to m, and its fractal dimension is less than or equal to $2m$, where m is given in* (56).

Remark. By the constants K_0, K, and A_i ($i = 1, \ldots, 11$) we find that when the $|\alpha_i|(i = 0, 2, 3, 4, 5)$ or $|\Omega|$ become large and the a_1 become small, then the upper bounds of the attractor dimensions become larger. This shows that with the increase of $|\alpha_i| (i = 0, 2, 3, 4, 5)$ or $|\Omega|$ and the decrease of a_1, the system becomes more complicated since the dimension of the attractor is one important character of complexity of the attractor.

In summary, we have shown that the 2D generalized Ginzburg–Landau equation, under periodic boundary condition and appropriate initial condition, has the maximal attractor with finite Hausdorff and fractal dimensions.

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